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Abstract Model and Controller Design for an Unstable Aircraft

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In this paper, we consider an unstable system with a time delay as an abstract model of an aircraft longitudinal motion for the short period. In our controller design we want both sensitivity reduction and robustness to certain unmodeled dynamics. Using standard arguments, these control objectives are formulated as an H^∞ optimal control problem. We show in detail how to compute the optimal controller for this distributed model (and not for a finite dimensional approximation). A numerical example is given to illustrate the computational procedure. Certain properties of the controller are discussed for this example.

I. Introduction

In recent years, a powerful control methodology known as H^∞ optimization has been developed to deal with a number of problems in robust control. Even though the mathematical theory associated with this methodology has been well worked out, there have not been many published nontrivial control examples to which the design techniques have actually been applied. In this paper, we address ourselves to such an example in which we will design a compensator for the control of an unstable aircraft model with a delay. In our design, we will work directly with the distributed model and not a finite dimensional approximation. Our technique is based on so-called *skew Toeplitz* theory, which allows one to design for (possibly) infinite dimensional systems.[1-5] These methods have already been applied to a distributed model of a flexible beam.[6]

One of the reasons to use distributed models in the controller design is that such infinite dimensional models may represent the dynamics of the physical system better than their finite dimensional approximates. On the other hand, in some cases infinite dimensional models that contain a few parameters are used for physical phenomenon that can otherwise be better explained by very high order finite dimensional models. Thus the economical representation of the system is another important reason why distributed models are used in practice. For example, in the abstract distributed model considered in this paper, there are only two parameters: a delay and an unstable pole.

The development of flight control requires a mathematical model of the aircraft flight dynamics. The components to be modeled include the dynamics of the aircraft, the control actuation device, the sensors, and the computer that implements the control law. It is expedient to make approximations that are known to be acceptable from the practice of flight control engineering. In Sec. II of this paper, we will review the aircraft dynamics and show that a transfer function with one unstable pole and a time delay forms an abstract model of the aircraft, for the purpose of controlling the longitudinal dynamics in the short period.

The abstract model is intended to be accurate for frequencies in the decade centered around the magnitude of the unstable pole. Thus low frequency dynamics are neglected, as well as high frequency dynamics. In the framework of the H^∞ control, it is possible to design a controller for the unmodeled dynamics. It is also possible to include a sensitivity reduction criteria to such a design. This is the key motivation to use H^∞ control. The details of how the control problem is formulated are given in Sec. III.

In Sec. IV we solve the H^∞ optimal control problem associated with this system. The procedure is illustrated with a numerical example in Sec. V. In this section we also discuss certain properties of the optimal controller. Finally, in Sec. VI we make some concluding remarks.

II. Abstract Model of an Unstable Aircraft

An aircraft rigid body has 6 degrees of freedom and thus 12 states. The forces and moments are independent of the direction the aircraft is traveling and the two horizontal position coordinates. The dependence of atmospheric properties on altitude can be neglected for the purposes of flight control design. This leaves eight states that can be further decomposed into two sets of four states because of the aircraft's plane of symmetry. The motions in the plane (two translations and one rotation) do not cause forces and moments in the out-of-plane motions (one translation and two rotations). The in-plane forces and moments caused by out-of-plane motions can be neglected. Thus the two sets of four states each are decoupled.

The in-plane motions are of interest for this paper. The four states model three degrees of freedom and several choices are used in practice. The specific state variables to be used in this paper are the magnitude of the velocity vector denoted by V, the flight-path angle denoted by γ, the angle of attack denoted by α, and the pitch angular rate denoted by q. These variables are shown in Fig. 1.

The in-plane motions, also known as the longitudinal motions, can be further decomposed by a time scale or frequency separation. The slowest motions or lowest frequency motions involve the exchange of gravitational potential energy and kinetic energy. This is an oscillatory motion that is lightly damped and has a frequency that is within a factor of three of g/V where g is the gravitational acceleration. This frequency is

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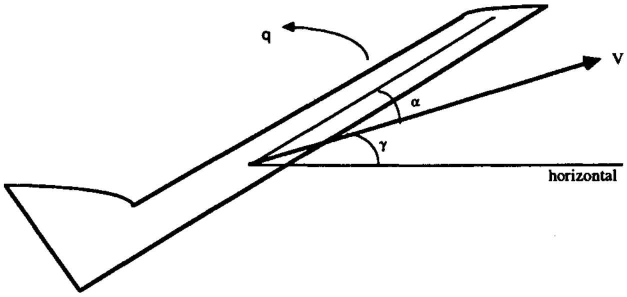

Fig. 1 Aircraft longitudinal states.

usually a decade below the frequencies of interest in this paper. The states associated with this so-called phugoid motion are V and γ and will be assumed to be constant in the following.

The high frequency longitudinal motion is known as the short period. In this special case, only the force normal to the flight path and the pitching moment will be of interest. The force is in the direction normal to the velocity and positive down. The moment is the pitching moment that is positive for nose up. The two variables that describe the motion of the aircraft are α and q.

The equations of motion are given by

$$mV\dot{\gamma} = mV(q - \dot{\alpha}) = L(\alpha,\delta,q) - mg\cos\gamma \tag{1}$$

$$I_{yy}\dot{q} = M_{\text{aero}}(\alpha,\delta,q) \tag{2}$$

where m is the mass of the aircraft, I_{yy} is the pitch axis moment of inertia, L is the aerodynamic lift force, M_{aero} is the aerodynamic pitching moment, and the propulsion force normal to the velocity vector and propulsion pitching moment have been neglected. The mass and moment of inertia are assumed to be constant. The aerodynamic terms depend on angle of attack, the control surface (horizontal tail) deflection denoted by δ, and the pitch rate.

For controls development, these equations are linearized about the straight and level equilibrium condition where $\dot{\alpha} = q = 0$ and $\gamma = 0$. The equilibrium condition is given by $(\alpha,\delta) = (\alpha_0,\delta_0)$ such that $L(\alpha_0,\delta_0,0) = mg$ and $M_{\text{aero}}(\alpha_0,\delta_0,0) = 0$.

After linearization the equations can be written

$$\dot{\alpha}_1 = Z_\alpha\alpha_1 + (1 + Z_q)q_1 + Z_\delta\delta_1 \tag{3}$$

$$\dot{q}_1 = M_\alpha\alpha_1 + M_q q_1 + M_\delta\delta_1 \tag{4}$$

where the subscript 1 denotes a perturbation from the equilibrium condition. That is, we have $\alpha = \alpha_0 + \alpha_1$, $q = q_1$, and $\delta = \delta_0 + \delta_1$.

The terms in the linearized equations arising from the partial derivatives of the aerodynamic functions are known as stability derivatives and are given by

$$Z_\alpha := \frac{-1}{mV}\frac{\partial L}{\partial \alpha}(\alpha_0,\delta_0,0)$$

and

$$M_\alpha := \frac{1}{I_{yy}}\frac{\partial M_{\text{aero}}}{\partial \alpha}(\alpha_0,\delta_0,0) \tag{5}$$

and similarly for Z_q, Z_δ, M_q, and M_δ.

Typically the terms Z_q and Z_δ are negligible and will be taken to be zero in the following. M_α is an important term because it determines longitudinal stability for the short period. This stability derivative can be positive or negative depending on the location of the aircraft center of mass. M_α increases as the center of mass moves aft. The stability derivatives Z_α and M_q are typically negative and provide damping. The characteristic equation is given by

$$s^2 - (Z_\alpha + M_q)s - M_\alpha + Z_\alpha M_q = 0 \tag{6}$$

thus if $M_\alpha > Z_\alpha M_q > 0$, the characteristic equation can be factored as

$$(s - a)(s + p) = 0 \tag{7}$$

where $a > 0$ is the positive root and $-p < 0$ is the negative root.

Typically the aircraft is controlled by measuring pitch rate with a gyroscope and rate of change of flight-path rate with an accelerometer. A common approach is to blend these two variables together to form a variable for feedback called y where

$$y = q + K\dot{\gamma} \tag{8}$$

and $K > 0$ is a constant and the subscript 1 (for linearization) was suppressed. In this case the transfer function from the control surface to the feedback variable is

$$\frac{y(s)}{\delta(s)} = \frac{M_\delta[s - Z_\alpha(1 + K)]}{(s - a)(s + p)} \tag{9}$$

Note that this transfer function has one negative zero because $Z_\alpha < 0$ and $K > 0$.

The next approximation is to assume a near pole-zero cancellation occurs in this transfer function; i.e., the blending variable K is such that $Z_\alpha(1 + K) \approx -p$. Then

$$\frac{y(s)}{\delta(s)} = \frac{M_\delta}{(s - a)} \tag{10}$$

The high frequency dynamics due to elasticity, actuators, sensors, computer, and zero order hold contribute effective time delays associated with the individual components. Combining all these into one transfer function, we get

$$P(s) = \frac{e^{-hs}}{\sigma s - 1} \tag{11}$$

as an abstract model of the aircraft for controlling unstable short period dynamics, where h is the total time delay in the feedback loop and $a := 1/\sigma$ is the unstable pole (we have normalized M_δ/a so that $|P(0)| = 1$). Note that only two parameters, σ and h, describe the model, which is infinite dimensional. It is difficult to control the aircraft if the product of the delay and the unstable pole (the ratio h/σ) is large. The X-29 aircraft at its most unstable flight condition has a product of unstable pole and effective time delay h/σ as large as 0.37, the other conditions being as much as a factor of 6 smaller. The meaning of "difficulty of the control" will be clear from the following discussion.

III. Control of the Unstable Aircraft Model

In the rest of this paper we will discuss the controller design for the abstract model, Eq. (11), of an unstable aircraft. The most important control objective is the stability of the closed-loop feedback control system. The controller that we are going to design is required to stabilize this open-loop unstable system. Also, the controller should take into account the unmodeled dynamics that are present in the actual system but do not appear in the model due to simplifications in the modeling. Another important specification in the control design is sensitivity reduction. We will consider all of these problems in our design. Under certain mild assumptions, these problems (namely, stabilization, robustness to unmodeled dynamics, and sensitivity reduction) can be put into one single optimality criteria, to be defined later.

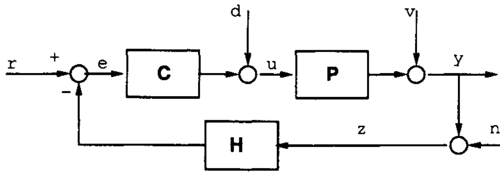

Fig. 2 Feedback system with externals inputs.

A. Stabilization

The feedback control system, including all possible external signals, is shown in Fig. 2. Here r, d, v, and n represent the reference input, actuator disturbance, output disturbance, and the measurement noise, respectively. These signals are external inputs to the closed-loop system. Internal signals in the system are the measured error e, command u, output y, and the noise corrupted output z. The system transfer functions are the plant model (unstable aircraft) P, the controller to be designed C, and the measurement device H.

We say that the closed-loop system, shown in Fig. 2, is *stable* if for all finite energy input signals $[r,d,v,n]$ the corresponding output signals $[z,e,u,y]$ have finite energy, and the maximum energy amplification from the inputs to the outputs is finite.

We make a simplifying assumption that $H(s) = 1$; i.e., the measurement device is perfect, except the noise n. Then, it is a well-known fact that the closed-loop system is stable if and only if all four transfer functions $S := (1 + PC)^{-1}$, $1 - S$, CS, and PS are in H^∞. [A function $F(s)$, $s \in C$ (complex numbers) is in H^∞ if it is analytic (no poles) in the right half plane and essentially bounded on the imaginary axis.] When a function F is essentially bounded on the imaginary axis, its infinity norm is defined as

$$\|F\|_\infty := \operatorname{ess}\sup_\omega |F(j\omega)| \tag{12}$$

Given the plant P, the set of all stabilizing controllers is characterized as follows. First write P as a ratio of two strongly coprime functions in H^∞, i.e., find functions P_n, $P_d \in H^\infty$ such that $P = P_n/P_d$ and

$$\inf_{\operatorname{Re} s>0} (|P_n(s)| + |P_d(s)|) > 0 \tag{13}$$

For our model $P(s) = e^{-hs}/(\sigma s - 1)$ we can choose

$$P_n(s) = e^{-hs}/(\sigma s + 1) \quad \text{and} \quad P_d(s) = (\sigma s - 1)/(\sigma s + 1) \tag{14}$$

The next step is to find X and Y in H^∞, solving the Bezout equation

$$P_n(s)X(s) + P_d(s)Y(s) = 1 \tag{15}$$

Since P_d has only one zero in the right half plane (i.e., pole of the unstable aircraft model P), there is only one interpolation condition for Y to exist in H^∞. That is the function $[1 - P_n(s)X(s)]$ should be zero at the zero of $P_d(s)$, i.e., at the pole $1/\sigma$ of the plant. Otherwise, $Y(s)$ has a pole at $1/\sigma$ and is not in H^∞. The solutions can be easily found:

$$X(s) = 2e^{h/\sigma}, \qquad Y(s) = \frac{(\sigma s + 1) - 2e^{h/\sigma}e^{-hs}}{(\sigma s - 1)} \tag{16}$$

Note that the numerator of $Y(s)$ becomes zero at $s = 1/\sigma$; hence $Y(s)$ does not have a pole in the right half plane. Then *all stabilizing controllers* are of the form (see, for example, Ref. 7)

$$C(s) = \frac{X(s) + P_d(s)Q(s)}{Y(s) - P_n(s)Q(s)} \tag{17}$$

where $Q \in H^\infty$ is the free parameter to be chosen according to the design specifications, other than stabilization of P.

B. Robustness to Unmodeled Dynamics

In the previous discussion, we assumed that the plant transfer function is given by P, then we characterized all controllers stabilizing this fixed model. However, as mentioned earlier, there are many simplifications made in obtaining the abstract model $P(s) = e^{-hs}/(\sigma s - 1)$. One must also take into account these simplifications in the controller design.

There are a number of ways to represent these unmodeled dynamics. The most common approach is to write the unmodeled dynamics as multiplicative, additive, or coprime factor perturbations of the nominal model P. Here we take the additive perturbation approach; that is, we assume that the true plant can be any transfer function of the form

$$G(s) = P(s) + W_2(s)\Delta(s) \tag{18}$$

where P is the nominal model, Δ is an arbitrary transfer function, representing the uncertainty in the plant, normalized to $\|\Delta\|_\infty < 1$, and W_2 is a fixed (usually known) transfer function representing the size and the distribution, over frequency, of the uncertainty. Without the loss of generality we can take W_2, $W_2^{-1} \in H^\infty$.

Considering the uncertainty in the plant, we require that the controller C, which is fixed, stabilizes not only the nominal model P but *all* possible plants of the form G. If a controller C meets this requirement, then we say that C *robustly stabilizes* the plant. Suppose that G and P have the same number of poles in the right half plane (in our aircraft model this number is 1). Then one can show that[8] a controller C robustly stabilizes the plant if and only if it stabilizes the nominal plant P and satisfies the robustness inequality

$$\|W_2C(1 + PC)^{-1}\|_\infty \le 1 \tag{19}$$

Note from the previous subsection that for our plant model the controller has to be of the form $C = (X + P_dQ)/(Y - P_nQ)$, $Q \in H^\infty$, where $P_n, P_d, X, Y \in H^\infty$ are given by Eqs. (14) and (16). In addition to the nominal model, Eq. (11), if an uncertainty description W_2 is given, then the robustness inequality (19) takes the form

$$\|W_2P_d(X + P_dQ)\|_\infty \le 1 \tag{20}$$

Thus a robustly stabilizing controller can be obtained by finding a transfer function $Q \in H^\infty$ satisfying Eq. (20).

Recall that for our aircraft model we have $P_d(s) = (\sigma_s - 1)/(\sigma s + 1)$, which is inner (all-pass) $X(s) = 2e^{h/\sigma}$, and we assume $W_2^{-1} \in H^\infty$. Therefore, a robustly stabilizing controller exists if an only if $1 \ge \lambda$, defined by

$$\lambda := \inf_{Q \in H^\infty} \|2e^{h/\sigma}W_2(s) - \left(\frac{\sigma s - 1}{\sigma s + 1}\right)Q\|_\infty \tag{21}$$

where we used Eqs. (14) and (16) in Eq. (20). This problem can be solved as follows.[9] Note that for any $F \in H^\infty$ we have $\|F\|_\infty \ge |F(1/\sigma)|$, so that

$$\|2e^{h/\sigma}W_2(s) - \left(\frac{\sigma s - 1}{\sigma s + 1}\right)Q\|_\infty \ge 2e^{h/\sigma}|W_2(1/\sigma)| \tag{22}$$

This is independent of $Q \in H^\infty$ because $Q(1/\sigma)$ must be finite. On the other hand,

$$Q_o(s) := 2e^{h/\sigma}\left\{\frac{W_2(s) - W_2(1/\sigma)}{(\sigma s - 1)/(\sigma s + 1)}\right\} \tag{23}$$

is in H^∞, and Q_o achieves the lower bound Eq. (22) when used in Eq. (21). Thus

$$\lambda = 2e^{h/\sigma}|W_2(1/\sigma)| \tag{24}$$

and so we can conclude that a robustly stabilizing controller exists if and only if the "size" of the plant uncertainty, at the unstable pole of the nominal model $|W_2(1/\sigma)|$, is less than $\frac{1}{2}e^{-h/\sigma}$. In other words, the "amount of uncertainty tolerated" by the nominal model decreases exponentially as the ratio h/σ increases. This illustrates the point mentioned at the end of Sec. II; that is, the ratio h/σ represents a difficult level for the control of the model $P(s) = e^{-hs}/(\sigma s - 1)$. Note that for the "easiest" case, $h/\sigma \to 0$, the maximum uncertainty $|W_2(1/\sigma)|$ cannot exceed 0.5; this number is 0.4709 and 0.3454 for h/σ equals 0.06 and 0.37, respectively.

C. Sensitivity Reduction

Up to this point we have only discussed problems of stabilization. Another important reason to use feedback control is to reduce the *system sensitivity*. The most common definition of system sensitivity (see, for example, Ref. 10) is the ratio of the percentage change in the closed-loop transfer function (from r to y) to the percentage change in the plant. Therefore, the system sensitivity is the function of $S(s) = [1 + P(s)C(s)]^{-1}$. There are many other equivalent definitions of the sensitivity, e.g., the transfer function from the output disturbance v to the output y, or the transfer function from the reference r to the error e, or the ratio of the percentage change in the output y to the percentage change in the plant, etc.

A natural control design specification is the sensitivity reduction, which can be stated as follows. Given a desired upper bound $W_d(s)$ for the sensitivity function (in general W_d, $W_d^{-1} \in H^\infty$), we want to find a stabilizing controller C such that

$$|[1 + G(j\omega)C(j\omega)]^{-1}| \leq |W_d(j\omega)| \qquad \text{for all } \omega \in \boldsymbol{R} \tag{25}$$

where G represents the true plant. Since the true plant is unknown and can be any transfer function of the form $G = P + W_2\Delta$, where P, W_2, and Δ are as before, we want a stabilizing C satisfying

$$|W_d^{-1}(j\omega)\{1 + [P(j\omega) + W_2(j\omega)\Delta(j\omega)]C(j\omega)\}^{-1}| \leq 1$$
$$\text{for all } \omega \in \boldsymbol{R} \tag{26}$$

and for all Δ satisfying the previous assumptions. This problem will be called the *robust performance* problem. Defining $W_1 := W_d^{-1}$ we see that Eq. (26) is equivalent to having

$$|W_1(1 + PC)^{-1}| \leq |1 + W_2\Delta C(1 + PC)^{-1}| \tag{27}$$

[we have dropped the dependence on $(j\omega)$ for notational convenience]. It is easy to see that condition (27) is satisfied if the following holds:

$$|W_1(1 + PC)^{-1}|^2 + W_2C(1 + PC)^{-1}|^2 \leq \tfrac{1}{2} \tag{28}$$

The left-hand side can be identified as follows: let $A(s) := W_1(s)[1 + P(s)C(s)]^{-1}$ and $B(s) := W_2(s)C(s)[1 + P(s)C(s)]^{-1}$. Note that if C stabilizes P, then A and B are H^∞ functions. The H^∞ norm of the vector valued function

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

is defined by

$$\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_\infty = \operatorname{ess\,sup}_{\omega} \sqrt{|A(j\omega)|^2 + B(j\omega)|^2} \tag{29}$$

Therefore, a sufficient condition for C to solve the robust performance problem is

$$\left\| \begin{bmatrix} W_1(1 + PC)^{-1} \\ W_2C(1 + PC)^{-1} \end{bmatrix} \right\|_\infty \leq \frac{1}{\sqrt{2}} \tag{30}$$

together with the stabilization of the nominal plant P. Hence, the robust performance problem is solvable if $\mu \leq 1/\sqrt{2}$ where μ is the optimal performance level corresponding to the following H^∞ optimal control problem:

$$\mu := \inf_{C:\ \text{stabilizing}} \left\| \begin{bmatrix} W_1(1 + PC)^{-1} \\ W_2C(1 + PC)^{-1} \end{bmatrix} \right\|_\infty \tag{31}$$

If $\mu \leq 1/\sqrt{2}$, then the H^∞ optimal controller C_{opt} solving the problem (31) also solves the robust performance problem. The previous problem is a "two-block" H^∞ problem; similarly the problem defined by Eq. (21) is a one-block problem. The meaning of a "block" should be clear from the previous definitions.

It is important to note that if μ is greater than $1/\sqrt{2}$, then one can define a new H^∞ optimal control problem by introducing new weighting functions $\hat{W}_i := kW_i$, $i = 1,2$. One can always choose k sufficiently small so that the new optimal performance $\hat{\mu}$, corresponding to the new H^∞ problem with $\hat{W}_1$, $\hat{W}_2$, P, satisfies the robust performance inequality $\hat{\mu} \leq 1/\sqrt{2}$. However, reducing the size of W_2 (multiplying it by a number $k < 1$) means that less uncertainty is tolerated by the nominal plant P. Similarly, reducing W_1 (multiplying it by $k < 1$) means increasing W_d, which is equivalent to increasing the desired system sensitivity by a factor of $1/k$.

Conversely, if $\mu < 1/\sqrt{2}$, then we can choose $k = 1/\mu\sqrt{2} > 1$ in the previous discussion and still satisfy the robust performance inequality. That is, if $\mu < 1/\sqrt{2}$, then the plant can tolerate an additional uncertainty, given by a factor of $1/\mu\sqrt{2}$.

IV. H^∞ Optimal Control of the Unstable Aircraft Model

We now study the H^∞ optimal control problem defined by Eq. (31) for our unstable aircraft model. The purpose is to find the optimal performance level μ and the corresponding optimal controller C_{opt}, given data P, W_1, and W_2. As we have discussed before, the smaller the μ, the better the performance of the closed-loop feedback control system.

The nominal plant model P is an infinite dimensional system. Therefore, the state space and algebraic Riccati equation method,[11-13] for the solution of this H^∞ control problem cannot be applied directly. An extension of the state space method to distributed systems is now available.[14] But this involves operator valued (infinite dimensional) Riccati equations, so the exact solutions are difficult to obtain. Therefore, the state space method is a difficult approach for our problem. There are only two parameters in the plant, namely, h and σ, and from this point of view the model is rather simple. Indeed, the skew Toeplitz theory developed for the H^∞ control of such distributed systems[1-6,15] allows us to compute μ and C_{opt} from a finite determinental formula. See also Refs. 16 and 17 for computation of μ in similar H^∞ control problems with delays. In Ref. 16 sensitivity minimization was considered only for stable plants, whereas in Ref. 17 only robust stabilization was addressed. Here we consider a mixture of these two problems (though Ref. 17 considers coprime factor uncertainty, whereas our uncertainty is additive).

The complexity of computations in the skew Toeplitz approach depends on the orders of the weights and the number of unstable poles of the plant and not on the dimension of the state space, which may be infinite. Since in our problem the plant is infinite dimensional with only one unstable pole, we apply the skew Toeplitz method here. In particular we use the results of Ref. 5 for our problem. To have relatively simpler computations, we consider low-order weights.

A. Reduction of the Two-Block Problem

In this section we will reduce the two-block problem of Eq. (31) to a one-block type problem by using standard techniques of H^∞ control. This way the complexity of the computations will be reduced. For the sake of completeness we show all of the details here. Recall that the controller for the plant $P(s) = e^{h/s}/(\sigma s - 1)$ is of the form

$$C(s) = \frac{X(s) + P_d(s)Q(s)}{Y(s) - P_n(s)Q(s)} \tag{32}$$

where $Q \in H^\infty$ is the free parameter, $P_d(s) = (\sigma s - 1)/(\sigma s + 1)$, $P_n(s) = e^{h/\sigma}/(\sigma s + 1)$, $X(s) = 2e^{h/\sigma}$, and $Y(s) = [1 - X(s)P_n(s)]/P_d(s)$. Let the weights be given as

$$W_1(s) := \frac{k}{\rho}\left(\frac{1 + \rho\tau s/\sqrt{1+\rho^2}}{1 + \tau s}\right)$$
$$W_2(s) := k\left(\frac{1 + \rho\tau s/\sqrt{1+\rho^2}}{1 + \sigma s}\right) \tag{33}$$

where k, ρ, and τ are free parameters to be chosen according to uncertainty description and sensitivity reduction conditions. See the next section for a choice and justification of these parameters. We consider first-order weights for simplicity. The theory of Ref. 5 can handle more general weights, but in that case the notation and computations are more complicated.

Substituting the expression for C in Eq. (32) (for the given plant and weights) in the definition

$$\mu = \inf_{C:\text{stabilizing}} \left\| \begin{bmatrix} W_1 S \\ W_2 CS \end{bmatrix} \right\|_\infty \tag{34}$$

$$\mu = \inf_{Q \in H^\infty} \left\| \begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} e^{-h/s} & 0 \\ 0 & \dfrac{\sigma s - 1}{\sigma s + 1} \end{bmatrix} \right.$$
$$\left. \times \begin{bmatrix} W_1/(\sigma s + 1) \\ -W_2 \end{bmatrix} \left[2e^{h/\sigma}\left(\frac{\sigma s - 1}{\sigma s + 1}\right)Q \right] \right\|_\infty \tag{35}$$

We now use the following well-known fact: if a 2×2 matrix $F_{si}(s)$, whose entries are in H^∞, satisfies $F_{si}(j\omega)^* F_{si}(j\omega) = I_{2\times 2}$ for all ω, then F_{si} is called *unitary* [here $F_{si}(j\omega)^*$ denotes the complex conjugate transpose of $F_{si}(j\omega)$, and $I_{2\times 2}$ is the 2×2 identity matrix]; furthermore, since F_{si} has the property that for any H^∞ functions G and H we have

$$\left\| F_{si} \begin{bmatrix} G \\ H \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} G \\ H \end{bmatrix} \right\|_\infty \tag{36}$$

Since the matrix

$$F_{si}(s) := \begin{bmatrix} e^{-hs} & 0 \\ 0 & \dfrac{\sigma s - 1}{\sigma s + 1} \end{bmatrix}$$

satisfies the previous property, we have

$$\mu = \inf_{Q \in H^\infty} \left\| \begin{bmatrix} W_1 e^{hs} \\ 0 \end{bmatrix} - \begin{bmatrix} W_1/(\sigma s + 1) \\ -W_2 \end{bmatrix} \right.$$
$$\left. \times \left[2e^{h/\sigma} + \left(\frac{\sigma s - 1}{\sigma s + 1}\right)Q \right] \right\|_\infty \tag{37}$$

It is easy to see that the weights are of the form

$$W_1(s) = \frac{k}{\rho}\left(\frac{1 + \sigma s}{1 + \tau s}\right)W(s), \qquad W_2(s) = kW(s) \tag{38}$$

where

$$W(s) := \left(\frac{1 + \rho\tau s/\sqrt{1+\rho^2}}{1 + \sigma s}\right) \tag{39}$$

Then we have

$$\mu = \frac{k}{\rho} \inf_{Q \in H^\infty} \left\| \left\{ \begin{bmatrix} \dfrac{1 + \sigma s}{1 + \tau s} e^{hs} \\ 0 \end{bmatrix} - \begin{bmatrix} (1 + \tau s)^{-1} \\ -\rho \end{bmatrix} \right. \right.$$
$$\left. \left. \times \left[2e^{h/\sigma}\left(\frac{\sigma s - 1}{\sigma s + 1}\right)Q \right] \right\} W(s) \right\|_\infty \tag{40}$$

We can find an inner/outer factorization for the matrix

$$V(s) := -\begin{bmatrix} (1 + \tau s)^{-1} \\ -\rho \end{bmatrix}$$

of the form $V(s) := F_i(s)F(s)$, where

$$F_i(s) = \begin{bmatrix} [F(s)(1 + \tau s)]^{-1} \\ -\rho F(s)^{-1} \end{bmatrix}$$
$$F(s) := \sqrt{1+\rho^2}\left(\frac{1 + \rho\tau s/\sqrt{1+\rho^2}}{1 + \tau s}\right) \tag{41}$$

Here $F_i \in H^\infty$ is inner, i.e., $F_i(j\omega)^* F_i(j\omega) = 1$, $F \in H^\infty$ is outer, i.e., has no poles or zeros in the right half plane. The outer factor $F(s)$ is computed for the spectral factor of $V(j\omega)^* V(j\omega)$ (see, for example, Ref. 9). Then, once F is computed, F_i is given by $F_i(s) = V(s) \cdot F(s)^{-1}$.

Now it is easy to check that the 2×2 matrix:

$$L(j\omega) := \begin{bmatrix} \left(1 - \dfrac{\rho}{\sqrt{1+\rho^2}} j\omega\right)^{-1} & -\rho \dfrac{1 - \tau j\omega}{1 - \rho j\omega/(\sqrt{(1+\rho^2)})} \\ \rho \dfrac{1 + \tau j\omega}{1 + \rho j\omega/(\sqrt{1+\rho^2})} & \left(1 + \dfrac{\rho}{\sqrt{1+\rho^2}} j\omega\right)^{-1} \end{bmatrix} \frac{1}{\sqrt{1+\rho^2}} \tag{42}$$

is unitary. Using the property (36) of L [i.e., multiplying the right-hand side of Eq. (40) by L], we reduce the problem to a one-block problem of the form

$$\mu_1 := \inf_{Q_1 \in H^\infty} \| W_0 - \hat{W}_0 M - M_d M Q_1 \|_\infty \tag{43}$$

where

$$W_0(s) := \frac{1}{\sqrt{1+\rho^2}} \frac{1}{1 + \tau s} \tag{44}$$

$$\hat{W}_0(s) := 2e^{h/\sigma}\sqrt{1+\rho^2}\,\frac{(1 + \rho\tau s/\sqrt{1+\rho^2})^2}{(1 + \tau s)(1 + \sigma s)} \tag{45}$$

$$M(s) := e^{-hs}\left(\frac{1 - \rho\tau s/\sqrt{1+\rho^2}}{1 + \rho\tau s/\sqrt{1+\rho^2}}\right)$$
$$M_d(s) := \left(\frac{\sigma s - 1}{\sigma s + 1}\right) \tag{46}$$

The two-block optimal performance μ can be computed from the one-block optimal performance μ_1; also Q and Q_1 has an invertible relationship:

$$\mu = \frac{k}{\rho}\sqrt{\mu_1^2 + \frac{\rho^2}{1+\rho^2}} \quad \text{and} \quad Q_1(s) = F(s)W(s)Q(s) \tag{47}$$

Note that F and W are invertible in H^∞.

B. Solution to the Reduced H^∞ Problem

The one-block H^∞ problem defined in Eq. (43) can be solved by using the formula given in Ref. 5. The first step is to transform the data from right half plane of the complex numbers C to the unit disc of C using a conformal map. That is, all the functions of s will be seen as functions of z, where $s \in C$, $\mathrm{Re}\, s \geq 0$, is replaced by $z \in C$, $|z| \leq 1$. Let the conformal map be

$$z = \frac{\sigma s - 1}{\sigma s + 1} \quad \text{and} \quad s = \frac{1}{\sigma}\frac{1+z}{1-z} \tag{48}$$

Then the transformed data are

$$w_0(z) := W_0(s)|_{s=\frac{1}{\sigma}\frac{1+z}{1-z}} = \frac{1}{(1+\tau/\sigma)\sqrt{1+\rho^2}}\left(\frac{1-z}{1-a_2 z}\right) \tag{49}$$

$$\hat{w}_0(z) := \hat{W}_0(s)|_{s=\frac{1}{\sigma}\frac{1+z}{1-z}}$$

$$= e^{h/\sigma}\sqrt{1+\rho^2}\,\frac{\left(1+\frac{\tau}{\sigma}\frac{\rho}{\sqrt{1+\rho^2}}\right)}{\left(1+\frac{\tau}{\sigma}\right)}\frac{(1-a_1 z)^2}{(1-a_2 z)} \tag{50}$$

$$m(z) := M(s)|_{s=\frac{1}{\sigma}\frac{1+z}{1-z}} = \left(\frac{a_1 - z}{1-a_1 z}\right)e^{\frac{h}{\sigma}\frac{z+1}{z-1}}$$

$$m_d(z) := M_d(s)|_{s=\frac{1}{\sigma}\frac{1+z}{1-z}} = z \tag{51}$$

where the real numbers a_1 and a_2 are given by

$$a_1 := \left[\frac{1-\frac{\tau}{\sigma}\frac{\rho}{\sqrt{1+\rho^2}}}{1+\frac{\tau}{\sigma}\frac{\rho}{\sqrt{1+\rho^2}}}\right], \qquad a_2 := \left[\frac{1-\frac{\tau}{\sigma}}{1+\frac{\tau}{\sigma}}\right] \tag{52}$$

Then the one-block H^∞ problem can be reduced to

$$\mu_2 := \inf_{q_2 \in H^\infty(D)}\left\|\left(\frac{1-z}{1-a_2 z}\right) - t\frac{(1-a_1 z)^2}{(1-a_2 z)}m(z) - zm(z)q_2(z)\right\|_\infty \tag{53}$$

[$H^\infty(D)$ is the usual Hardy space H^∞ defined on the unit disc D] where

$$t = (1+\rho^2)\left(1+\frac{\tau}{\sigma}\frac{\rho}{\sqrt{1+\rho^2}}\right)^2 e^{h/\sigma} \tag{54}$$

There is an invertible relationship between Q_1 and q_2, and between μ_1 and μ_2:

$$\mu_1 = \frac{\mu_2}{(1+\tau/\sigma)\sqrt{1+\rho^2}}$$

$$Q_1(s) = \frac{q_2(z)}{(1+\tau/\sigma)\sqrt{1+\rho^2}}\bigg|_{z=\frac{\sigma s-1}{\sigma s+1}} \tag{55}$$

We are now ready to solve our one-block problem. But before we go into the mathematical details, we would like to present some background material on the shift operators on Hilbert spaces. We refer the interested reader to Ref. 18 for an introduction to these concepts.

1. Shift and Compressed Shift Operators

The second Hardy space H^2 is the space of functions $f(z)$: $D \to C$ having an analytic power series expansion

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad \text{with} \quad \sum_{k=0}^{\infty} |f_k|^2 < \infty \tag{56}$$

The canonical shift operator, denoted by S, can be represented by multiplication by z, i.e., shift S acting on an element $f \in H^2$ generates a new element in H^2 given by $g(z) = zf(z)$. Note that $g(z) = 0 + f_0 z^1 + f_1 z^2 + \ldots$.

Recall $m(z)$ defined earlier; it is an inner function [i.e., it is in H^∞ and satisfies $m(e^{j\theta})^*(me^{j\theta}) = 1$], and so is the function $zm(z)$. We can define two subspaces of H^2 associated with $m(z)$ and $zm(z)$ as follows. Let

$$m(z)H^2 := \{m(z)f(z) : f \in H^2\}$$

$$zm(z)H^2 := \{zm(z)f(z) : f \in H^2\} \tag{57}$$

Then the subspaces of H^2 that are orthogonal complements of $zm(z)H^2$ and $m(z)H^2$ are denoted by $H[zm(z)]$ and $H[m(z)]$, respectively. This means that any function $f \in H^2$ has orthogonal decompositions $f(z) = g(z) + zm(z)h(z)$ and $f(z) = \hat{g}(z) + m(z)\hat{h}(z)$, where $g \in H[zm(z)]$, $\hat{g}(z) \in H[m(z)]$, and h, $\hat{h} \in H^2$. An important consequence of the previous definitions is that any element u of $H[zm(z)]$ can be written in terms of orthogonal components $u(z) = p(z) + m(z)q$ where $p \in H[m(z)]$ and $q \in C$.

Also, definitions of subspace $H[zm(z)]$ and $H[m(z)]$ imply that the functions $z^{-1}m(z^{-1})g(z)$ and $m(z^{-1})\hat{g}(z)$ are of the form

$$z^{-1}m(z^{-1})g(z) = \sum_{i=1}^{\infty}\phi_{-i}z^{-i}$$

and

$$m(z^{-1})\hat{g}(z) = \sum_{i=1}^{\infty}\hat{\phi}_{-i}z^{-i} \tag{58}$$

where the right-hand sides converge outside the unit circle for some coefficients ϕ_i, $\hat{\phi}_i$, $\in C$, $i = 1,2,\ldots$, such that

$$\sum_{i=1}^{\infty}|\phi_i|^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty}|\hat{\phi}_i|^2 < \infty \tag{59}$$

Since $H[zm(z)]$ is a subspace of H^2, we can define an orthogonal projection operator $P_{H[zm(z)]} : H^2 \to H[zm(z)]$. This operator acting on an arbitrary element f of H^2 generates $P_{H[zm(z)]}f = g$, the $H[zm(z)]$ component of $f(z) = g(z) + zm(z)h(z)$. Using the orthogonal projection, we can define the *compressed shift* operator, denoted by T, associated with $H[zm(z)]$ as follows: $T : H[zm(z)] \to H[zm(z)]$, for $g \in H[zm(z)]Tg = P_{H[zm(z)]}Sg$.

The *adjoint* of the shift operator is denoted by S^*: for $f \in H^2$, $S^*f := e$ where $e \in H^2$ and $e(z) = z^{-1}[f(z) - f(0)]$. Similarly we denote the adjoint of the compressed shift as T^*, and the action of T^* is the same as the action of S^* except that T^* acts on elements from $H[zm(z)]$ and generates elements in $H[zm(z)]$.

An important point to note is that the spaces H^2, $H[zm(z)]$ and $H[m(z)]$ are infinite dimensional. So the operators defined on these spaces can be seen as infinite size matrices acting on infinite size vectors represented by the coefficients of the functions of the type, say $f \in H^2$, $f(z) = f_0 + f_1 z^1 + f_2 z^2 + \ldots$. However, for our problem it is not practical to use infinite size vector and matrix notation; in the following discussion, we will only need to know the *action* of T and T^* on elements of $H[zm(z)]$.

2. Computation of μ_2 and the Optimal Controller

Using the commutant lifting theorem,[19] it can be shown (see, for example, Ref. 3) that μ_2 is the norm of the operator $A : H[zm(z)] \to H[zm(z)]$ defined in terms of T by

$$A = (I - T)(I - a_2 T)^{-1} - t(I - a_1 T)^2 m(T)(I - a_2 T)^{-1} \quad (60)$$

The norm of A is the largest of the two positive quantities, the essential norm (see, for example, Refs. 2 and 5 for a precise definition), and the largest singular value of A. In our problem, we do not have to worry about the essential norm because it is zero. Therefore, the quantity μ_2 is the largest singular value of A if and only if there exists a nonzero singular vector $x \in H[zm(z)]$ such that

$$(A^*A - \rho_2^2 I)x = 0 \quad (61)$$

where A^* is the adjoint of A. The previous equation can be written explicitly in terms of action of T and T^*: There exists $0 \neq x \in H[zm(z)]$ satisfying the previous singular value singular vector equation if and only if there is $0 \neq u \in H[zm(z)]$ satisfying

$$\{[I - T^* - tm(T)^*(I - a_1 T^*)^2][I - T - t(I - a_1 T)^2 m(T)] - \rho_2^2 (I - a_2 T^*)(I - a_2 T)\}u = 0 \quad (62)$$

where $u = (I - a_2 T)^{-1}x$, or $x = (I - a_2 T)u$ [since $|a_2| < 1$, the operator $(I - a_2 T)$ is invertible and this invertible relationship is used in Eqs. (61) and (60) to obtain Eq. (62)]. All of the terms in Eq. (62) can be computed explicitly by decomposing u into its orthogonal components: $u(z) = p(z) + m(z)q$ where $p \in H[m(z)]$, and $q \in C$. The details can be found in Ref. 5. To give an idea how these expressions look, we give the following terms:

$$(I - T)u = (1 - z)p(z) + m(z)q \quad (63)$$

$$m(T)^*(I - T)u = q - \gamma_{-1} \quad (64)$$

$$m(T)u = m(z)\gamma_0 \quad (65)$$

$$m(T)^* m(T)(I - a_1 T)^2 u = \gamma_0 \quad (66)$$

$$Tm(z)\gamma_0 = 0 \quad (67)$$

$$T^* m(z)\gamma_0 = z^{-1}[m(z) - m(0)]\gamma_0 \quad (68)$$

where $\gamma_0 = u(0)$, and γ_{-1} comes from $m(z^{-1})p(z) =: \gamma_{-1}z^{-1} + \gamma_{-1}z^{-2} \ldots$.

Using the previous identities, we obtain three equations in three variables γ_{-1}, γ_0, $q \in C$ for ρ_2 to be a singular value of A, i.e., for a nonzero $u \in H[zm(z)]$ to satisfy Eq. (62). Again, these equations are obtained by taking the orthogonal projections of Eq. (62) onto the subspaces $H[m(z)]$ and C. In particular the projection of Eq. (62) on C implies that

$$q = \frac{[1 - a_2\rho_2^2 - tm(0)]\gamma_{-1} + t[1 - tm(0)]\gamma_0}{[1 - \rho_2^2 - tm(0)]} \quad (69)$$

On the other hand, the projection of Eq. (62) on $H[m(z)]$ gives us

$$p(z) = \frac{R_0(z)\gamma_0 + R_{-1}(z)\gamma_{-1} + R_q(z)q}{(a_2\rho_2^2 - 1)\left\{z^2 - \left[\dfrac{2 - \rho_2^2(1 + a_2^2)}{1 - a_2\rho_2^2}\right]z + 1\right\}} \quad (70)$$

where

$$R_0(z) := \rho_2^2 a_2 - 1 - t^2 z + t[zm(z) - m(z) + m(0)] \quad (71)$$

$$R_{-1}(z) := -tz \quad (72)$$

$$R_q(z) := (\rho_2^2 - 1)zm(z) + (1 - a_2\rho_2^2)m(z) + tz \quad (73)$$

Note that by definition p is the $H[m(z)]$ component of $u \in H[zm(z)]$. Equation (70) gives us two interpolation conditions for p to be a nonzero element of $H[m(z)]$; that is, the numerator of Eq. (70) should vanish at the roots of its denominator polynomial, which is second order. This gives us two equations expressed in terms of ρ^2 and the constants γ_{-1}, γ_0, and q. Moreover, when these two equations and Eq. (69) hold, we can define a nonzero u in $H[zm(z)]$ satisfying the singular vector equation (62). See Ref. 5 for details. These three equations can be written as

$$R_{\rho_2}\Phi = 0 \quad (74)$$

where

$$R_{\rho_2} = \begin{bmatrix} R_{1,\gamma-1} & R_{1,\gamma 0} & R_{1,q} \\ R_{2,\gamma-1} & R_{2,\gamma 0} & R_{2,q} \\ R_{3,\gamma-1} & R_{3,\gamma 0} & R_{3,q} \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \gamma_{-1} \\ \gamma_0 \\ q \end{bmatrix} \quad (75)$$

and the entries of R_{ρ_2} are given by

$$R_{1,\gamma-1} = -tz_1 \quad (76)$$

$$R_{1,\gamma 0} = \rho_2^2 a_2 - 1 - t^2 z_1 + t[z_1 m(z_1) - m(z_1) + m(0)] \quad (77)$$

$$R_{1,q} = (\rho_2^2 - 1)z_1 m(z_1) + (1 - a_2\rho_2^2)m(z_1) + tz_1 \quad (78)$$

$$R_{2,\gamma-1} = -tz_2 \quad (79)$$

$$R_{2,\gamma 0} = \rho_2^2 a_2 - 1 - t^2 z_2 + t[z_2 m(z_2) - m(z_2) + m(0) \quad (80)$$

$$R_{2,q} = (\rho_2^2 - 1)z_2 m(z_2) + (1 - a_2\rho_2^2)m(z_2) + tz_2 \quad (81)$$

$$R_{3,\gamma-1} = 1 - a_2\rho_2^2 - tm(0) \quad (82)$$

$$R_{3,\gamma 0} = t[1 - tm(0)] \quad (83)$$

$$R_{3,q} = \rho_2^2 - 1 + tm(0) \quad (84)$$

and z_1 and z_2 are the roots of the polynomial in the denominator of Eq. (70):

$$z_1 = \frac{1 - \rho_2^2[(1 + a_2^2)/2]}{1 - \rho_2^2 a_2} + \frac{\rho_2(1 - a_2)}{|1 - \rho_2^2 a_2|}\sqrt{\rho_2^2\left(\frac{1 + a_2}{2}\right)^2 - 1} \quad (85)$$

$$z_2 = \frac{1 - \rho_2^2[(1 + a_2^2)/2]}{1 - \rho_2^2 a_2} - \frac{\rho_2(1 - a_2)}{|1 - \rho_2^2 a_2|}\sqrt{\rho_2^2\left(\frac{1 + a_2}{2}\right)^2 - 1} \quad (86)$$

Let us summarize the previous discussion. We have seen the following:

1) μ_2 is the norm of A.

2) The norm of A, hence μ_2, is the largest singular value of A.

3) Any positive number ρ_2 is a singular value of A if and only if there exists a nonzero $u \in H[zm(z)]$ such that Eq. (62) holds.

4) There exists a nonzero $u \in H[zm(z)]$ satisfying Eq. (62) if and only if there exists a nonzero Φ, with entries in C, such that $R_{\rho 2}\Phi = 0$; such *nonzero* Φ exists if and only if the matrix $R_{\rho 2}$ is singular.

Hence, from Secs. I–IV we conclude that the quantity μ_2 can be computed as the largest value of ρ_2 that makes the matrix $R_{\rho 2}$ singular; i.e., μ_2 can be found by plotting the minimum singular value of $R_{\rho 2}$ vs ρ_2. The largest value ρ_2 where this plot has a zero is μ_2 (see Ref. 20 to get an idea how these plots look). Once μ_2 is found, we can also compute a nonzero vector Φ^o satisfying

$$R_{\mu 2}\Phi^o = 0 \quad (87)$$

Then a singular vector $u^o(z) = p^o(z) + m(z)q^o$, with $p^o \in H[m(z)]$ and $q^o \in C$, can be obtained from the entries γ^o_{-1}, γ^o_0, and q^o of Φ^o. Going back from the commutant lifting theorem, we can find an optimal $q_{2,\mathrm{opt}}(z) \in H^\infty$ yielding the performance μ_2. We refer the reader to Refs. 3 and 5 for the proofs of these facts. After all these manipulations, we obtain the following formula for $q_{2,\mathrm{opt}}$:

$$q_{2,\mathrm{opt}}(z) = \frac{(1-a_2z)t\gamma^o_0 - z(1-a_2)q^o - t(1-a_1z)^2[(1-a_2z)p^o(z) + m(z)q^o]}{z(1-a_2z)[(1-a_2z)p^o(z) + m(z)q^o]} \tag{88}$$

where

$$p^o(z) = \frac{p^o_0(z)\gamma^o_0 + p^o_{-1}(z)\gamma^o_{-1} + p^o_q(z)q^o}{(a_2\mu_2^2 - 1)\left[z^2 - \left(\dfrac{2-\mu_2^2(1+a_2^2)}{1-a_2\mu_2^2}\right)z + 1\right]} \tag{89}$$

$$p^o_0(z) := \mu_2^2a_2 - 1 - t^2z + t[zm(z) - m(z) + m(0)] \tag{90}$$

$$p^o_{-1}(z) := -tz \tag{91}$$

$$p^o_q(z) := (\mu_2^2 - 1)zm(z) + (1 - a_2\mu_2^2)m(z) + tz \tag{92}$$

Finally, using the relationship between $q_{2,\mathrm{opt}}(z)$ and $C_{\mathrm{opt}}(s)$, we see that the optimal controller is given by

$$C_{\mathrm{opt}}(s) = \left(\frac{\sigma s-1}{\sigma s+1}\right)\left\{\frac{2e^{h/\sigma} + \dfrac{\sigma s-1}{\sigma s+1}Q_{\mathrm{opt}}(s)}{1 - \dfrac{e^{-hs}}{\sigma s+1}\left[2e^{h/\sigma} + \dfrac{\sigma s-1}{\sigma s+1}Q_{\mathrm{opt}}(s)\right]}\right\} \tag{93}$$

where

$$Q_{\mathrm{opt}}(s) = \left.\frac{F(s)^{-1}W(s)^{-1}q_{2,\mathrm{opt}}(z)}{(1+\tau/\sigma)\sqrt{1+\rho^2}}\right|_{z=\frac{\sigma s-1}{\sigma s+1}} \tag{94}$$

and $F(s)$, $W(s)$, and $q_{2,\mathrm{opt}}(z)$ are given earlier.

We remark once again that first two of the three equations represented by Eq. (87) guarantees that $p^o(z)$, defined by Eq. (89), is in $H[m(z)]$. Also the third equation guarantees that the numerator of $q_{2,\mathrm{opt}}(z)$ vanishes at $z = 0$. This makes sure that at $s = 1/\sigma$ the denominator of the second term on the right-hand side of Eq. (93) is zero; hence this cancels the zero $1/\sigma$ of the first term. This means that $C_{\mathrm{opt}}(s)$ *does not have a zero at* $1/\sigma$. Thus, there is no unstable pole zero cancellation in the $C_{\mathrm{opt}}(s)P(s)$ product. In fact, by design, this optimal controller stabilizes the nominal plant and achieves the optimal H^∞ performance μ.

V. Numerical Example

In this section we demonstrate the previous theory by illustrating a numerical example. The procedure described in Sec. IV is a control algorithm that can easily be implemented on the computer. In fact, we wrote a MatLab program to compute the optimal performance μ and the optimal controller C_{opt} for the *two-block* H^∞ problem considered earlier.

Recall that the plant is given by $P(s) = e^{-hs}/(\sigma s - 1)$, and weights are given as

$$W_1(s) = \frac{k}{\rho}\left(\frac{1+\rho\tau s/\sqrt{1+\rho^2}}{1+\tau s}\right)$$

$$W_2(s) = \frac{k}{\rho}\left(\frac{1+\rho\tau s/\sqrt{1+\rho^2}}{1+\sigma s}\right) \tag{95}$$

The inputs to the computer program are the parameters h, σ, k, ρ, and τ, and the outputs are the optimal performance level μ, and the singular vector Φ^o satisfying $R_{\mu 2}\Phi^o = 0$. Using the formulas (88), (93), and (94), we see that μ and Φ^o are sufficient to determine the optimal controller C_{opt}. But the closed-form expression of the controller is rather complicated. On the other hand, we can evaluate $C_{\mathrm{opt}}(s)$ for any given s; in particular we will give here the Bode plots for $C_{\mathrm{opt}}(j\omega)$, which is also a part of our MatLab program.

To demonstrate what happens to μ and C_{opt} as we increase the difficulty level h/σ, we will fix the weights and the parameter σ (say $\sigma = 1$) and vary h between 0.06 and 0.37. We choose $k = 0.2$, $\rho = 0.1$, and $\tau = 10$, corresponding magnitude plots for W_1 and W_2 are given in Fig. 3. This choice of W_1 means that the sensitivity function is to be reduced by a factor of 2, in the frequency range lower than one decade below the corner

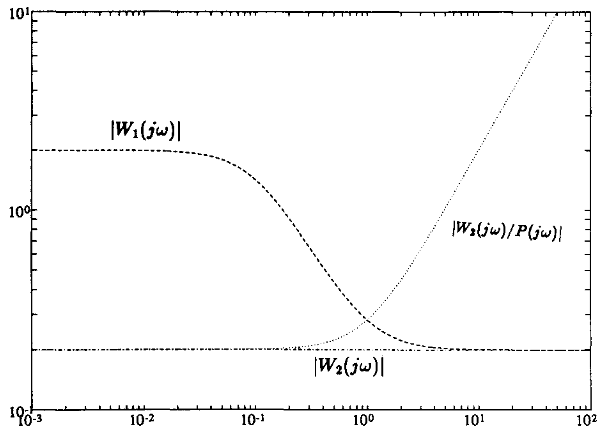

Fig. 3 Weights and the relative uncertainty.

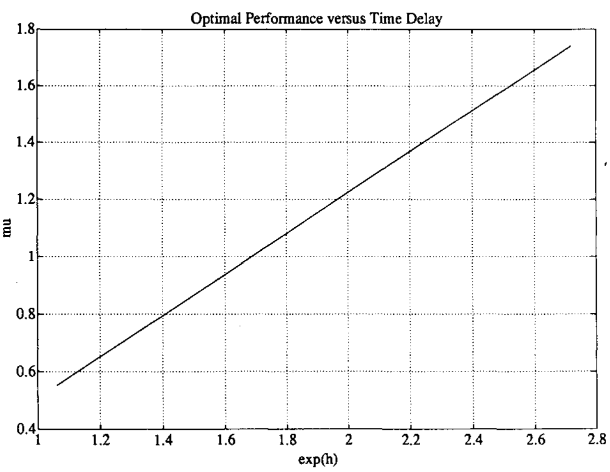

Fig. 4 μ vs $e^{h/\sigma}$, $\sigma = 1$.

Table 1 Optimal performance vs time delay

h	0.001	0.01	0.06	0.15	0.20	0.25	0.37	0.55	0.85	1.0	1.5	2.0	3.0	5.0
μ	0.5092	0.5157	0.5529	0.6244	0.6669	0.7116	0.8283	1.0326	1.4685	1.74	3.02	5.135	14.38	107.74

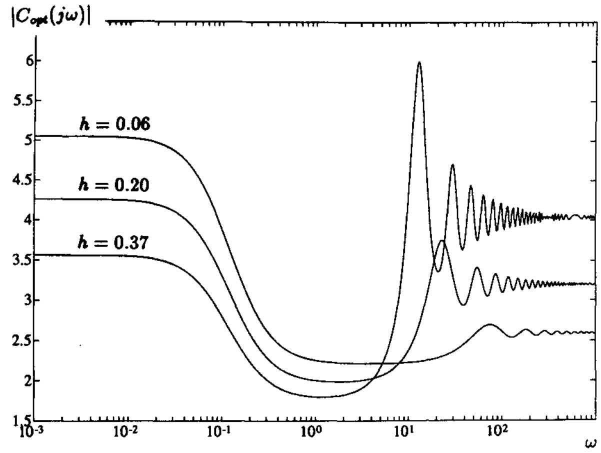

Fig. 5 $|C_{\mathrm{opt}}(j\omega)|$ vs ω.

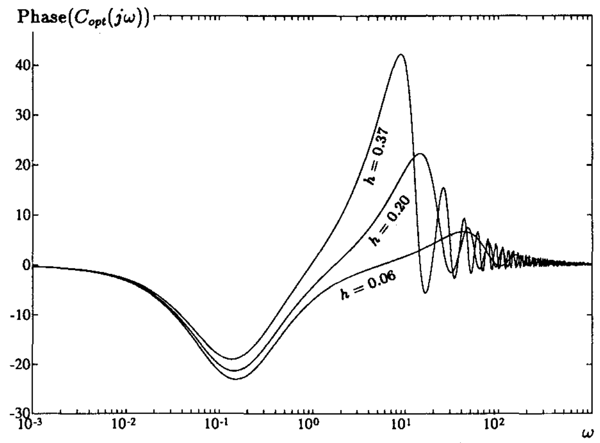

Fig. 6 Phase $[C_{\mathrm{opt}}(j\omega)]$ vs ω.

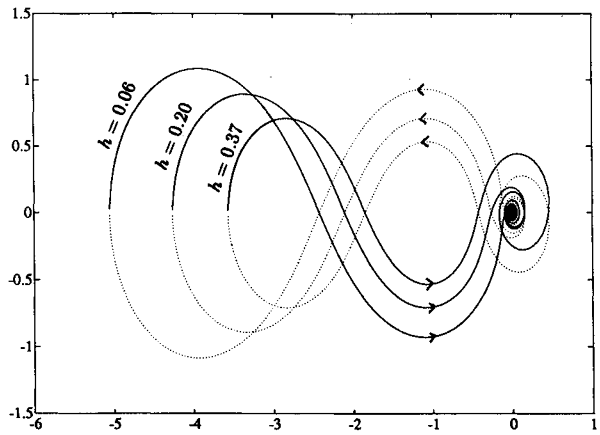

Fig. 7 Nyquist plot $P(j\omega)C_{\mathrm{opt}}(j\omega)$.

frequency of $|P(j\omega)|$. The choice of W_2 suggests that the size of the additive uncertainty is 0.2; this is more or less constant throughout all frequencies. Relative to the magnitude of the plant, the uncertainty is about 20% up to the corner frequency of P. After this frequency, the magnitude of relative uncertainty, $|[W_2(j\omega)]/[P(j\omega)]|$, increases 20 dB per decade. This is a typical choice for the weights in the two-block H^∞ design (see, for example, Ref. 21).

The optimal performance μ for this numerical problem is computed for several different values of the delay h. The results are given in Table 1. To get a better idea about the behavior of μ with respect to the changes in h, we have plotted some of these points in Table 1 (see Fig. 4). We observe an exponential relationship between μ and h. We also give the

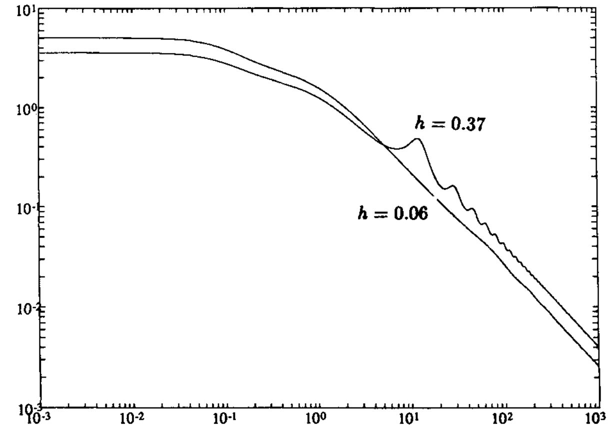

Fig. 8 Magnitude plot $|P(j\omega)C_{\mathrm{opt}}(j\omega)|$.

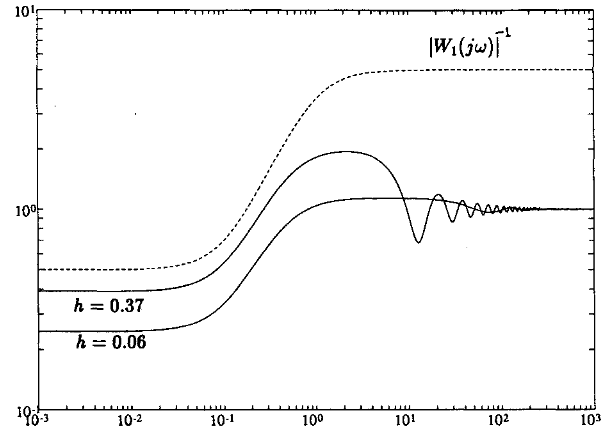

Fig. 9 Optimal sensitivity $|[1+P(j\omega)C_{\mathrm{opt}}(j\omega)]^{-1}|$.

magnitude (Fig. 5) and phase (Fig. 6) plots of the optimal controller for values of h equal to 0.06, 0.20, and 0.37. An important point to note is that the optimal controller $C_{\mathrm{opt}}(s)$ contains delay terms e^{-hs}; therefore, it is infinite dimensional. On the other hand, the Bode plots (Figs. 5 and 6) show that $C_{\mathrm{opt}}(j\omega)$ is continuous on the imaginary axis, which means that it is uniformly approximable by a finite dimensional controller (see, for example, Refs. 22 and 23). We observe from these figures that the approximation becomes more difficult (i.e., we need higher order approximations) as h increases. Once again this illustrates that h/σ is a sensible difficulty measure of the controller design, for the abstract aircraft model. The issue of approximating this optimal controller is discussed in detail in Refs. 24 and 25.

The Nyquist plots for these controllers are also provided; see Fig. 7 where the dotted line corresponds to the values of $P(j\omega)C_{\mathrm{opt}}(j\omega)$ for negative values of ω, the solid line corresponds to positive values of ω, and the arrows show the increasing direction of ω. Note that the critical point -1 is encircled once in the counterclockwise direction. Since by design C_{opt} stabilizes the nominal plant, and P has one unstable pole, we deduce from this Nyquist plot that the optimal controller is stable. The magnitude plots for the optimal open-loop transfer function $P(j\omega)C_{\mathrm{opt}}(j\omega)$ are shown in Fig. 8 for $h = 0.37$ and 0.06.

We also provide the magnitude plots (Fig. 9), for the optimal sensitivity function $S_{\mathrm{opt}} = (1 + PC_{\mathrm{opt}})^{-1}$ for the same values of h. We observe that

$$|S_{\mathrm{opt}}(j\omega)| < |W_1(j\omega)^{-1}| = |W_d(j\omega)| \tag{96}$$

This shows that the desired performance is achieved for the nominal plant P.

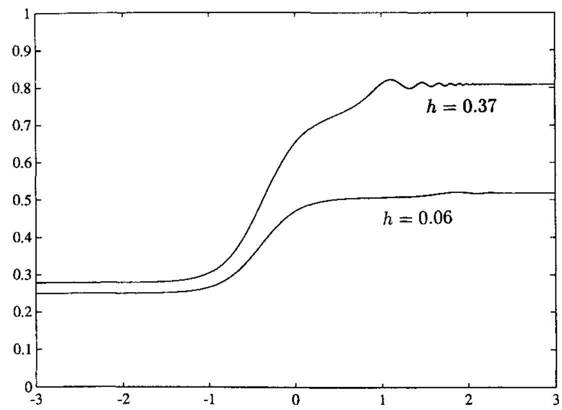

Fig. 10 Robust stability test: $|W_2(j\omega)C_{\rm opt}(j\omega)[1+P(j\omega)C_{\rm opt}(j\omega)]^{-1}|$.

Similarly, Fig. 10 shows that

$$|W_2(j\omega)C_{\rm opt}(j\omega)[1+P(j\omega)C_{\rm opt}(j\omega)]^{-1}| \le 1 \tag{97}$$

for all ω. In other words, robustness inequality (19) is satisfied. This illustrates robust stability. Furthermore, Fig. 10 gives an idea how much more uncertainty can be tolerated by this optimal design: the uncertainty level $|W_2(j\omega)|$ can be increased by a factor of $\{|W_2(j\omega)C_{\rm opt}(j\omega)[1+P(j\omega)C_{\rm opt}(j\omega)]^{-1}|\}^{-1}$ while preserving robust stability.

VI. Concluding Remarks

In this paper, we have applied an H^∞ optimal control design to an unstable system with delay that appears in aircraft control. The H^∞ problem considered here was a mixture of the problems of sensitivity reduction and robustness to additive unstructured perturbations in the plant model. The skew Toeplitz methods that we employed gave the optimal compensator whose performance we were able to explicitly study as a function of the various system parameters. The advantage of using such a method is that it allows one to work directly with the delay and, moreover, gives explicit numerical results in a straightforward rigorous way without the need for any approximations.

An important point to mention is that the optimal controller obtained here is a rather complicated expression [see Eqs. (88-93)], containing delay terms, which make $C_{\rm opt}$ infinite dimensional. There are essentially two ways to obtain finite dimensional suboptimal controllers, stabilizing P and achieving a performance close to μ. The first one is to approximate the optimal controller and check stability and performance. The second method is to approximate the plant, obtain the corresponding optimal controller, which will be finite dimensional, then check stability and compare the performance with μ. Although the first method looks difficult (because it approximates a complicated infinite dimensional system: $C_{\rm opt}$), it is numerically feasible to do such approximations. See Refs. 23 and 9 for the details of this method applied to the problem considered in this paper. The first part (approximating the plant) of the second method is easier, but to compare the performance with the optimum, one needs to *know,* or *estimate,* the optimum μ, which requires an H^∞ *optimal* control method applicable to unstable delay systems. We are planning to present a comparison of these two techniques in Ref. 20.

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